PHYSICAL REVIEW E **77**, 061103 (2008)

Intrinsic fractality of classic shot noise

Iddo Eliazar*
Department of Technology Management, Holon Institute of Technology, P.O. Box 305, Holon 58102, Israel
(Received 10 February 2008; published 4 June 2008)

We revisit the classic shot noise of Campbell and Schottky—a stochastic process governed by Ornstein-Uhlenbeck dynamics driven by a Poissonian noise. Exploring the order statistics of the shot magnitudes composing its stationary noise level, we show that classic shot noise is intrinsically fractal. This fractality is manifested by (**i**) intrinsic Paretian and scale-invariant structures; (**ii**) an intrinsic power-law scaling; (**iii**) an intrinsic statistical resilience to random power-law perturbations.

DOI: 10.1103/PhysRevE.77.061103 PACS number(s): 05.40.Ca, 05.45.Df, 02.50.−r

I. INTRODUCTION

Shot noise, pioneered by Campbell [1,2] and Schottky [3] at the beginning of the twentieth century, is a cornerstone model in statistical physics. In its "classic" form, a shot noise system is abstractly described as follows: A random source emits 'shots' of unit magnitude stochastically in time, the shot emissions following a Poisson process with constant intensity (η). Once emitted, the shots decay exponentially at a constant rate (κ). The shot noise level at time t—henceforth denoted $\xi(t)$—is the aggregate of all shot magnitudes present at time t (originating from shots emitted up to time t).

Denoting by $\{\tau_i\}_i$ the shots' emission epochs—which form a Poisson process with intensity η—the shot noise level at time t is given by

$$\xi(t) = \sum_{\tau_i \le t} \exp(-\kappa(t-\tau_i)). \tag{1}$$

The dynamics of the shot noise process $(\xi(t))_t$, in turn, are governed by the Ornstein-Uhlenbeck stochastic differential equation

$$\dot{\xi}(t) = -\kappa\xi(t) + \dot{N}(t), \tag{2}$$

driven by the Poisson noise $\dot{N}(t)=\Sigma_i \delta(t-\tau_i)$ [$\delta(\cdot)$ denoting Dirac's delta function]. [1]

The classic shot noise process $(\xi(t))_t$ and its generalizations were explored extensively by researchers. Rice conducted a comprehensive statistical analysis and asymptotic analysis of shot noise [4,5]. Gilbert and Pollak investigated the stationary probability distribution of shot noise [6]. Lowen and Teich introduced fractal shot noise [7], and power-law shot noise [8]. Eliazar and Klafter introduced nonlinear shot noise [9], and studied the simultaneous display of the Noah and Joseph effects by general shot noise processes [10]. (This short list of references is far from being exhaustive.)

In this research we revisit the classic shot noise described above and explore the order statistics of the shot magnitudes composing its stationary noise level. Our study unveils the *intrinsic fractality* of classic shot noise, manifested by (i) Paretian structure (Sec. III); (ii) scale-invariant structure (Sec. IV); (iii) power-law scaling (Sec. V); (iv) statistical resilience to random power-law perturbations (Sec. VI).

Classic shot noise thus turns out to be an elemental stochastic example of how most "benign" and "well-behaved" inputs—a simple Poissonian shot inflow coupled linearly with a simple exponential decay—can give rise to an intricate statistically fractal structure.

We begin with some preliminaries (Sec. II), and thereafter present our research (Sec. III–VI). Throughout the manuscript the acronym "i.i.d." stands for "independent and identically distributed," and the sign $\overset{\text{Law}}{=}$ denotes equality in law (in distribution) of random objects.

II. PRELIMINARIES

We consider a classic shot noise process $(\xi(t))_{-\infty<t<\infty}$ initiated at time $t=-\infty$. This process is both Markov and stationary—with mean η/κ, variance $\eta/2\kappa$, and autocorrelation function $\exp\{-\kappa|t|\}$ $(-\infty<t<\infty)$ [11,12].

The process's stationary noise level is governed by

(i) Laplace transform

$$L(\theta) = \exp\left\{-\frac{\eta}{\kappa}\int_0^1 \frac{1-\exp\{-\theta x\}}{x}dx\right\} \tag{3}$$

($\theta \ge 0$);

(ii) harmonic cumulant sequence

$$C_m = \frac{\eta}{\kappa}\frac{1}{m} \tag{4}$$

($m=1,2,\ldots$). [2]

With no loss of generality, consider the shot noise level $\xi(0)$ at time $t=0$. This shot noise level is the aggregate of all shot magnitudes originating from shots arriving up to time $t=0$. Denote by $\mathcal{O}_n$ the size of the nth largest shot magnitude

*eliazar@post.tau.ac.il

[1] Equation (1) follows from Eq. (2) by straightforward differentiation with respect to the temporal variable t.

[2] The first-order cumulant C_1 and the second-order cumulant C_2 are, respectively, the mean and variance.

$(n=1,2,\ldots)$ composing the shot noise level $\xi(0)$. Labeling the shots' emission epochs up to time $t=0$ by $\cdots<\tau_{-3}<\tau_{-2}<\tau_{-1}<0$, Eq. (1) implies that $\mathcal{O}_n=\exp(\kappa\tau_{-n})$ $(n=1,2,\ldots)$ and $\xi(0)=\Sigma_{n=1}^{\infty}\mathcal{O}_n$.

The sequence of shot magnitudes $\{\mathcal{O}_n\}_{n=1}^{\infty}$ is ordered decreasingly $1>\mathcal{O}_1>\mathcal{O}_2>\mathcal{O}_3>\cdots$, and is henceforth referred to as the sequence of *order statistics* of the shot magnitudes composing the stationary shot noise level (sampled, with no loss of generality, at time $t=0$). This sequence:

(i) forms an *inhomogeneous Poisson process* scattered on the unit interval $(0,1)$ with *harmonic intensity*

$$\lambda(x)=\frac{\eta}{\kappa}\frac{1}{x} \quad (5)$$

$(0<x<1)$;

(ii) admits the stochastic representation

$$\{\mathcal{O}_n\}_{n=1}^{\infty}\overset{\text{Law}}{=}\{(\mathcal{U}_1\cdots\mathcal{U}_n)^{\kappa/\eta}\}_{n=1}^{\infty}, \quad (6)$$

where $\{\mathcal{U}_n\}_{n=1}^{\infty}$ is a sequence of i.i.d. random variables distributed uniformly on the unit interval $(0,1)$.

Equations (3)–(5) stem from counterpart results regarding general shot noise processes (see, for example, [13]). The proof of Eq. (6) is given in the Appendix (Sec. 1). The Laplace transform $L(\theta)$ of Eq. (3) and the harmonic intensity $\lambda(x)$ of Eq. (5) are connected via

$$L(\theta)=\exp\left\{-\int_0^1(1-\exp\{-\theta x\})\lambda(x)dx\right\}, \quad (7)$$

$(\theta\geq 0)$. Equation (7)—a special case of Campbell's theorem (a key result in the theory of Poisson processes [14])—is the analytic manifestation of the connection $\xi(0)=\Sigma_{n=1}^{\infty}\mathcal{O}_n$.

III. INTRINSIC PARETIAN STRUCTURE

Consider the sequence of consecutive ratios of the shot noise order statistics $\{\mathcal{O}_n\}_{n=1}^{\infty}$—namely, the random sequence $\{\mathcal{O}_n/\mathcal{O}_{n+1}\}_{n=1}^{\infty}$. The ratio $\mathcal{O}_n/\mathcal{O}_{n+1}$—taking values in the ray $(1,\infty)$—measures the size of the nth order statistic with respect to the size of the $(n+1)$th order statistic.

The stochastic representation of Eq. (6) implies that the consecutive ratios $\mathcal{O}_n/\mathcal{O}_{n+1}$ $(n=1,2,\ldots)$ are *independent* random variables governed by a common *Pareto law* with exponent η/κ. Specifically:

$$\left\{\frac{\mathcal{O}_n}{\mathcal{O}_{n+1}}\right\}_{n=1}^{\infty}\overset{\text{Law}}{=}\{\mathcal{P}_n\}_{n=1}^{\infty}, \quad (8)$$

where $\{\mathcal{P}_n\}_{n=1}^{\infty}$ is a sequence of i.i.d. random variables taking values in the ray $(1,\infty)$ and governed by the probability distribution

$$\Pr(\mathcal{P}_n>x)=\left(\frac{1}{x}\right)^{\eta/\kappa} \quad (9)$$

$(x\geq 1)$.

[In the derivation of Eq. (8) from Eq. (6) we used the fact that if $\mathcal{U}$ is a random variable uniformly distributed on the unit interval $(0,1)$, then the random variable $\mathcal{U}^{-1/\alpha}$ is Pareto distributed with exponent α: $\Pr(\mathcal{U}^{-1/\alpha}>x)=x^{-\alpha}$ $(x\geq 1$; α being an arbitrary positive exponent).]

Pareto's law, discovered in the context of human income [15], describes a power-law connection between positive-valued measurements and their occurrence frequencies. Empirically, Pareto's law is ubiquitously observed across the Sciences—see [16] and references therein. Theoretically, Pareto's law manifests *statistical fractality*—see Chap. 38 in [17].

The intrinsic Paretian structure of the shot noise order statistics $\{\mathcal{O}_n\}_{n=1}^{\infty}$, as shall be demonstrated in the following section, is indeed a manifestation of statistical fractality—a scale-invariant structure of the order statistics $\{\mathcal{O}_n\}_{n=1}^{\infty}$.

The ubiquity of Pareto's law in diverse real world systems motivated researchers to seek mechanisms capable of generating this law. Examples include preferential attachment (Yule process [18], Simon's model [19]), self-organized criticality ([20] and references therein), and the oligarchy mechanism (for the universal generation of Pareto's law with integer-valued exponents [21]). (For detailed reviews of "Pareto-generating mechanisms" the readers are referred to Chap. 14 in [22], and [16,23].)

As shown here, classic shot noise is yet another mechanism which naturally generates Pareto's law from most "benign" and "well-behaved" building blocks: A simple Poisson process coupled linearly with a simple exponential decay.

IV. INTRINSIC SCALE-INVARIANT STRUCTURE

Consider the order statistics $\mathcal{O}_{m+1}>\mathcal{O}_{m+2}>\mathcal{O}_{m+3}>\cdots$, normalized with respect to the size of the m^{th} order statistic $\mathcal{O}_m$. This yields the normalized sequence of order statistics $\{\mathcal{O}_{m+n}/\mathcal{O}_m\}_{n=1}^{\infty}$.

The stochastic representation of Eq. (6) implies that the normalized sequences $\{\mathcal{O}_{m+n}/\mathcal{O}_m\}_{n=1}^{\infty}$ $(m=1,2,\ldots)$ are all equal, in law, to the "original" sequence of order statistics $\{\mathcal{O}_n\}_{n=1}^{\infty}$. Namely:

$$\left\{\frac{\mathcal{O}_{m+n}}{\mathcal{O}_m}\right\}_{n=1}^{\infty}\overset{\text{Law}}{=}\{\mathcal{O}_n\}_{n=1}^{\infty} \quad (10)$$

$(m=1,2,\ldots)$.

Equation (10) implies that the statistical structure of the shot noise order statistics $\{\mathcal{O}_n\}_{n=1}^{\infty}$ is *self-similar*. Any order statistic "looking down" on the order statistics smaller than itself—while normalizing their sizes with respect to his own size—will observe the *scale-invariant* statistical structure $\{\mathcal{O}_n\}_{n=1}^{\infty}$ (the "scale" here being the order of the order statistic "looking down").

Is this scale-invariance *unique*? Do other shot noise systems admit the intrinsic scale-invariant structure of Eq. (10)?

To answer these questions consider a general shot noise system "fed" by shots with random i.i.d. magnitudes of generic size M, and whose shot-decay pattern is governed by the decay-dynamics $\dot{x}=-F(x)$ [13]. [3] In this general setting,

[3] The function $F(x)$ $(x>0)$ needs to be positive valued, and such that the ordinary differential equation $\dot{x}=-F(x)$ is well defined and has solutions decaying to zero.

the statistical structure of the shot noise order statistics is scale invariant in the sense of Eq. (10) if and only if: (**i**) the generic magnitude size M takes values in the unit interval $(0,1]$; and (**ii**) the function $F(x)$ and the probability distribution $\Pr(M>x)$ are related via

$$\frac{\Pr(M>x)}{F(x)} = \frac{c}{x}, \quad (11)$$

($0<x<1$; c being an arbitrary positive constant). The proof of Eq. (11) is given in the Appendix (Sec. 2).

In the case of classic shot noise, the magnitudes are deterministic and of unit-size ($M\equiv 1$), and the decay dynamics are linear ($\dot{x}=-\kappa x$)—hence Eq. (11) is indeed satisfied. Equation (11) further implies that: (**i**) in the case of unit-size shot magnitudes scale invariance in the sense of Eq. (10) holds if and only if the decay-dynamics are linear; (**ii**) in the case of linear decay-dynamics scale invariance in the sense of Eq. (10) holds if and only if the shot magnitudes are of unit-size. Thus, as long as the shot decay is exponential (i.e., governed by linear decay dynamics) then classic shot noise is the *unique* shot noise system whose stationary order statistics structure is scale invariant.

V. INTRINSIC POWER-LAW SCALING

Consider a multiplicative change of intensity $\eta\mapsto\eta'=p\eta$—the factor p being an arbitrary positive parameter—in the Poisson process $(N(t))_{-\infty<t<\infty}$ driving the shot noise process $(\xi(t))_{-\infty<t<\infty}$. Equation (6) implies that the corresponding sequence of shot noise order statistics—denote it by $\{\mathcal{O}_n'\}_{n=1}^{\infty}$—admits the stochastic representation

$$\{\mathcal{O}_n'\}_{n=1}^{\infty} \overset{\text{Law}}{=} \{(\mathcal{U}_1\cdots\mathcal{U}_n)^{\kappa/\eta'}\}_{n=1}^{\infty}, \quad (12)$$

[$\{\mathcal{U}_n\}_{n=1}^{\infty}$ being a sequence of i.i.d. random variables distributed uniformly on the unit interval $(0,1)$].

On the other hand, consider the power-law transformation $x\mapsto y=x^{1/p}$ ($0<x,y<1$) applied to each and every of the shot noise order statistics $\{\mathcal{O}_n\}_{n=1}^{\infty}$—yielding the transformed sequence $\{\mathcal{O}_n^{1/p}\}_{n=1}^{\infty}$. Since $\eta'=p\eta$, Eq. (6) implies that the transformed sequence $\{\mathcal{O}_n^{1/p}\}_{n=1}^{\infty}$ admits the stochastic representation

$$\{\mathcal{O}_n^{1/p}\}_{n=1}^{\infty} \overset{\text{Law}}{=} \{(\mathcal{U}_1\cdots\mathcal{U}_n)^{\kappa/\eta'}\}_{n=1}^{\infty}, \quad (13)$$

[$\{\mathcal{U}_n\}_{n=1}^{\infty}$ being a sequence of i.i.d. random variables distributed uniformly on the unit interval $(0,1)$].

Equations (12) and (13) imply that the sequences $\{\mathcal{O}_n'\}_{n=1}^{\infty}$ and $\{\mathcal{O}_n^{1/p}\}_{n=1}^{\infty}$ are equal in law. Hence, the *linear* change of Poissonian intensity $\eta\mapsto p\eta$ is statistically equivalent to the *nonlinear* transformation of the order statistics $\mathcal{O}_n\mapsto\mathcal{O}_n^{1/p}$ ($n=1,2,\ldots$). The *intrinsic scaling* of classic shot noise is thus *power law*.

VI. INTRINSIC STATISTICAL RESILIENCE TO RANDOM POWER-LAW PERTURBATIONS

In light of the previous section, consider a random power-law perturbation of the form $x\mapsto y=x^{1/\zeta}$ ($0<x,y<1$)—the exponent ζ being a randomly chosen positive parameter—applied, independently, to each and every of the shot noise order statistics $\{\mathcal{O}_n\}_{n=1}^{\infty}$. Specifically, consider the random power-law perturbation

$$\{\mathcal{O}_n\}_{n=1}^{\infty} \mapsto \{\mathcal{O}_n^{1/\zeta_n}\}_{n=1}^{\infty}, \quad (14)$$

where $\{\zeta_n\}_{n=1}^{\infty}$ are i.i.d. copies of ζ—an arbitrary positive valued random variable with finite mean (denoted $\langle\zeta\rangle$).

The random power-law perturbation of Eq. (14) is *order breaking*: While the input sequence $\{\mathcal{O}_n\}_{n=1}^{\infty}$ is ordered decreasingly ($1>\mathcal{O}_1>\mathcal{O}_2>\mathcal{O}_3>\cdots$), the output sequence $\{\mathcal{O}_n^{1/\zeta_n}\}_{n=1}^{\infty}$ is unordered. Yet, the output sequence $\{\mathcal{O}_n^{1/\zeta_n}\}_{n=1}^{\infty}$ is an *inhomogeneous Poisson process* scattered on the unit interval $(0,1)$ with *harmonic intensity*

$$\lambda_{\zeta}(x) = \langle\zeta\rangle\frac{\eta}{\kappa}\frac{1}{x}, \quad (15)$$

($0<x<1$). The proof of Eq. (15) is given in the Appendix (Sec. 3).

Recalling that the sequence of order statistics $\{\mathcal{O}_n\}_{n=1}^{\infty}$ is an inhomogeneous Poisson process scattered on the unit interval $(0,1)$ with harmonic intensity $\lambda(x)=(\eta/\kappa)/x$ [Eq. (5)], and exploiting the results of the previous section, we conclude that the *nonlinear*, *random*, and *order-breaking* perturbation of Eq. (14) is statistically equivalent to: (**i**) the *linear* and *deterministic* change of Poissonian intensity $\eta\mapsto\eta'=\langle\zeta\rangle\eta$; (**ii**) the *deterministic* and *order-preserving* nonlinear transformation

$$\{\mathcal{O}_n\}_{n=1}^{\infty} \mapsto \{\mathcal{O}_n^{1/\langle\zeta\rangle}\}_{n=1}^{\infty}. \quad (16)$$

The fact that the transformation $\lambda(x)\mapsto\lambda_{\zeta}(x)=\langle\zeta\rangle\lambda(x)$ preserves the harmonic functional structure $(1/x)$—while affecting only the amplitude of harmonic intensity (replacing the amplitude η/κ by the amplitude $\langle\zeta\rangle\eta/\kappa$)—means that the stationary structure of classic shot noise is *statistically resilient* to random power-law perturbations.

VII. CONCLUSIONS

We revisited the classic shot noise of Campbell and Schottky in which shots of unit magnitude arrive following a temporal Poisson process and decay exponentially. The stationary shot noise level is the aggregate of countably many shot magnitudes scattered randomly along the unit interval.

A statistical analysis of the order statistics of the aforementioned stationary shot magnitudes unveiled the fractal nature of classic shot noise, manifested by (**i**) intrinsic Paretian and scale-invariant statistical structures; (**ii**) an intrinsic power-law scaling; (**iii**) an intrinsic statistical resilience to random power-law perturbations. It was further shown that (**iv**) classic shot noise is a natural mechanism for the generation of Pareto's law; (**v**) the intrinsic scale-invariant structure of classic shot noise is unique amongst all shot noise systems with exponential shot decay.

APPENDIX

1. Proof of Eq. (6)

Recall that the shots' emission epochs up to time $t=0$ were labeled $\cdots < \tau_{-3} < \tau_{-2} < \tau_{-1} < 0$, and set $\tau_0 = 0$ (τ_0 is *not* an emission epoch—we set $\tau_0=0$ only for the convenience of notation). Since the shots' emission epochs follow a Poisson process with intensity η, the random differences $\{\tau_{-n} - \tau_{-n-1}\}_{n=0}^{\infty}$ are i.i.d. and exponentially distributed with mean $1/\eta$. In other words, we have

$$\{\tau_{-n} - \tau_{-n-1}\}_{n=0}^{\infty} \overset{\text{Law}}{=} \{\mathcal{E}_n/\eta\}_{n=1}^{\infty}, \tag{A1}$$

where $\{\mathcal{E}_n\}_{n=1}^{\infty}$ is a sequence of i.i.d. random variables distributed exponentially with unit mean. Hence

$$\{\tau_{-n}\}_{n=1}^{\infty} \overset{\text{Law}}{=} \left\{ -\frac{1}{\eta}(\mathcal{E}_1 + \cdots + \mathcal{E}_n) \right\}_{n=1}^{\infty}. \tag{A2}$$

Since $\mathcal{O}_n = \exp(\kappa\tau_{-n})$ $(n=1,2,\ldots)$ Eq. (A2) implies that

$$\{\mathcal{O}_n\}_{n=1}^{\infty} = \{\exp(\kappa\tau_{-n})\}_{n=1}^{\infty} \overset{\text{Law}}{=} \left\{ \exp\left(-\frac{\kappa}{\eta}(\mathcal{E}_1 + \cdots + \mathcal{E}_n) \right) \right\}_{n=1}^{\infty}. \tag{A3}$$

And, setting $\mathcal{U}_n = \exp(-\mathcal{E}_n)$ $(n=1,2,\ldots)$, Eq. (A3) yields the stochastic representation:

$$\{\mathcal{O}_n\}_{n=1}^{\infty} \overset{\text{Law}}{=} \left\{ \exp\left(-\frac{\kappa}{\eta}(\mathcal{E}_1 + \cdots + \mathcal{E}_n) \right) \right\}_{n=1}^{\infty} \overset{\text{Law}}{=} \left\{ (\mathcal{U}_1 \cdots \mathcal{U}_n)^{\kappa/\eta} \right\}_{n=1}^{\infty}. \tag{A4}$$

Finally, the fact that the random variables $\{\mathcal{E}_n\}_{n=1}^{\infty}$ are i.i.d. and exponentially distributed with unit mean implies that the random variables $\{\mathcal{U}_n\}_{n=1}^{\infty}$ are i.i.d. and uniformly distributed on the unit interval (0,1)—concluding the proof.

2. Proof of Eq. (11)

We split the proof into two parts.

Part I

Consider an arbitrary inhomogeneous Poisson process scattered on the range $(0,l)$ with intensity $r(x)$ $(x>0)$. The range's upper bound l may be either finite $(l<\infty)$ or infinite $(l=\infty)$.

The Poisson process has infinitely many points if and only if its intensity is nonintegrable ($\int_0^l r(x)dx = \infty$). The points of the Poisson process can be ordered decreasingly—by a random sequence $l > X_1 > X_2 > X_3 > \cdots$—if and only if its intensity is integrable at the upper bound l (namely, $\int_t^l r(x)dx < \infty$ for all $0<t<l$).

In what follows, we consider the Poisson process to have infinitely many points which can be ordered decreasingly. This holds if and only if the function $R(t) = \int_t^l r(x)dx$ $(0<t<l)$ decreases monotonically from the limit $\lim_{t\to 0} R(t) = \infty$ to the limit $\lim_{t\to l} R(t) = 0$, in which case the sequence of order statistics $l > X_1 > X_2 > X_3 > \cdots$ admits the stochastic representation

$$\{X_n\}_{n=1}^{\infty} \overset{\text{Law}}{=} \{R^{-1}(\mathcal{E}_1 + \cdots + \mathcal{E}_n)\}_{n=1}^{\infty}, \tag{A5}$$

where [24] (**i**) the function $R^{-1}(x)$ $(x>0)$ is the inverse of the function $R(t)$ $(0<t<l)$; (**ii**) $\{\mathcal{E}_n\}_{n=1}^{\infty}$ is a sequence of i.i.d. random variables distributed exponentially with unit mean.

The sequence of order statistics $\{X_n\}_{n=1}^{\infty}$ is scale-invariant—in the sense of Sec. IV—if the following counterpart of Eq. (10) holds:

$$\left\{ \frac{X_{m+n}}{X_m} \right\}_{n=1}^{\infty} \overset{\text{Law}}{=} \{X_n\}_{n=1}^{\infty}, \tag{A6}$$

$(m=1,2,\ldots)$. Substituting the stochastic representation of Eq. (A5) into Eq. (A6) further implies that the following equality (in law) must hold:

$$\left\{ \frac{R^{-1}(\mathcal{E}_1 + \cdots + \mathcal{E}_{m+n})}{R^{-1}(\mathcal{E}_1 + \cdots + \mathcal{E}_m)} \right\}_{n=1}^{\infty} \overset{\text{Law}}{=} \{R^{-1}(\mathcal{E}_1 + \cdots + \mathcal{E}_n)\}_{n=1}^{\infty}, \tag{A7}$$

$(m=1,2,\ldots)$.

Now, Eq. (A7) holds if and only if the inverse function $R^{-1}(x)$ satisfies $R^{-1}(x+y) = R^{-1}(x)R^{-1}(y)$ $(x,y>0)$. Hence, the inverse function $R^{-1}(x)$ is an exponential: $R^{-1}(x) = \exp(\rho x)$ $(x>0)$. This, in turn, implies that the function $R(t)$ is a logarithm: $R(t) = \frac{1}{\rho}\ln(t)$ $(0<t<1)$. Since the function $R(t)$ is monotone decreasing (from infinity to zero) we obtain that it admits the logarithmic form $R(t) = -a\ln(t)$ $(0<t<1)$, where a is an arbitrary positive amplitude. The corresponding intensity $r(x)$, in turn, admits the harmonic form $r(x) = a/x$ $(0<x<1)$. Our conclusion is thus as follows:

Consider an arbitrary inhomogeneous Poisson process scattered on the range $(0,l)$ with intensity $r(x)$ $(x>0)$, whose points can be ordered decreasingly by an infinite sequence of order statistics. The order statistics are scale invariant—in the sense of Sec. IV—if and only if: (**i**) the range is the unit interval $(l=1)$; and, (**ii**) the intensity is harmonic $r(x) = a/x$ $(0<x<1$; a being an arbitrary positive amplitude).

Part II

Consider a general shot noise system "fed" by shot magnitudes taking arbitrary values in the range $(0,l]$, as follows.

A random source emits "shots" stochastically in time. The shot emissions follow a Poisson process with constant intensity (η), and the shot magnitudes are i.i.d. realizations of a random magnitude (M) taking values in the range $(0,l]$. Once emitted, the shots decay patterns are governed by the decay-dynamics $\dot{x} = -F(x)$ [where the function $F(x)$ $(x>0)$ is positive valued, and such that the ordinary differential equation $\dot{x} = -F(x)$ is well defined and has solutions decaying to zero] [13].

As in the case of classic shot noise, the stationary shot noise level is the aggregate of countably many shot magnitudes—albeit scattered randomly along the range $(0,l)$ [rather than along the unit interval $(0,1)$]. Moreover [13], the aforementioned "stationary shot magnitudes" form an inhomogeneous Poisson process scattered on the range $(0,l)$ with intensity

$$r(x) = \eta \frac{\Pr(M > x)}{F(x)}, \tag{A8}$$

$(0<x<l)$. [In the case of linear decay dynamics Eq. (A8) stems also from results regarding Lévy-driven Ornstein-Uhlenbeck processes [25].]

Now, the analysis carried out in Part I asserts that the stationary structure of the shot noise under consideration is scale invariant—in the sense of Sec. IV—if and only if (**i**) the range is the unit interval ($l=1$); and, (**ii**) the intensity $r(x)$ of Eq. (A8) is harmonic: $r(x)=a/x$ ($0<x<1$; a being an arbitrary positive amplitude).

Hence, for the general shot noise system considered in this part, we conclude that the statistical structure of the shot noise order statistics is scale invariant in the sense of Eq. (10) if and only if (**i**) the generic magnitude size M takes values in the unit interval $(0,1]$; and (**ii**) the function $F(x)$ and the probability distribution $\Pr(M>x)$ are related via

$$\frac{\Pr(M > x)}{F(x)} = \frac{c}{x}, \tag{A9}$$

($0<x<1$; c being an arbitrary positive constant).

3. Proof of Eq. (15)

Fix an arbitrary positive valued random variable ζ with finite mean, and let $\psi_\zeta(u)$ ($u>0$) denote ζ's probability density function. Consider the random map $x \mapsto y = x^{1/\zeta}$ ($0<x,y<1$). Given the input point x, it is straightforward to deduce that the output point $y=x^{1/\zeta}$ is a random variable governed by the probability density function

$$\Phi(x;y) = \psi_\zeta\left(\frac{\ln(x)}{\ln(y)}\right)\frac{-\ln(x)}{(\ln(y))^2 y}, \tag{A10}$$

$(0<x,y<1)$.

We turn now to the random power-law perturbation of Eq. (14). The "displacement theorem" of the theory of Poisson processes ([14], Sec. 5.5) asserts that (**i**) the right-hand side of Eq. (14) is an inhomogeneous Poisson process scattered on the unit interval $(0,1)$; (**ii**) the connection between the Poissonian intensity $\lambda(x)$ of the left-hand side of Eq. (14), and the Poissonian intensity $\lambda_\zeta(y)$ of the right-hand side of Eq. (14), is given by

$$\lambda_\zeta(y) = \int_0^1 \Phi(x;y)\lambda(x)dx, \tag{A11}$$

$(0<y<1)$.

Substituting Eq. (A10) into Eq. (A11), while using the change of variables $u=\ln(x)/\ln(y)$, further yields

$$\lambda_\zeta(y) = \int_0^\infty [y^{u-1}\lambda(y^u)][u\psi_\zeta(u)]du, \tag{A12}$$

$(0<y<1)$. Finally, since $\lambda(x)=(\eta/\kappa)/x$ [Eq. (5)], and since the random variable ζ is of finite mean ($\langle\zeta\rangle=\int_0^\infty u\psi_\zeta(u)du<\infty$), Eq. (A12) implies that

$$\lambda_\zeta(y) = \langle\zeta\rangle\frac{\eta}{\kappa}\frac{1}{y}, \tag{A13}$$

$(0<y<1)$—concluding the proof.

[1] N. Campbell, Proc. Cambridge Philos. Soc. **15**, 117 (1909).
[2] N. Campbell, Proc. Cambridge Philos. Soc. **15**, 310 (1909).
[3] W. Schottky, Ann. Phys. **57**, 541 (1918).
[4] S. O. Rice, Bell Syst. Tech. J. **23**, 282 (1944).
[5] S. O. Rice, Bell Syst. Tech. J. **24**, 146 (1945).
[6] E. N. Gilbert and H. O. Pollak, Bell Syst. Tech. J. **39**, 333 (1960).
[7] S. B. Lowen and M. C. Teich, Phys. Rev. Lett. **63**, 1755 (1989).
[8] S. B. Lowen and M. C. Teich, IEEE Trans. Inf. Theory **36**, 1302 (1990).
[9] I. Eliazar and J. Klafter, in *Proceedings of the National Academy of Sciences* (National Academy of Sciences, Washington, D. C., 2005), Vol. 102, p. 13779.
[10] I. Eliazar and J. Klafter, Phys. Rev. E **75**, 031108 (2007).
[11] E. Parzen, *Stochastic Processes* (SIAM, Philadelphia, 1999).
[12] S. B. Lowen and M. C. Teich, *Fractal-based Point Processes* (Wiley, New York, 2005).
[13] I. Eliazar and J. Klafter, Physica A **360**, 227 (2006).
[14] J. F. C. Kingman, *Poisson Processes* (Oxford University Press, Oxford, 1993).
[15] V. Pareto, *Cours d'Économie Politique* (Droz, Geneva, 1896).
[16] M. E. J. Newman, Contemp. Phys. **46**, 323 (2005).
[17] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1982).
[18] G. Yule, Philos. Trans. R. Soc. London, Ser. B **213**, 21 (1925).
[19] H. A. Simon, Biometrika **42**, 425 (1955).
[20] P. Bak, *How Nature Works: The Science of Self-organized Criticality* (Copernicus, New York, 1996).
[21] I. Eliazar and J. Klafter, Physica A **377**, 53 (2007).
[22] D. Sornette, *Critical Phenomena in Natural Sciences* (Springer, Berlin, 2000).
[23] M. Mitzenmacher, Internet Math. **1**, 226 (2004).
[24] I. Eliazar, Prob. Eng. Info. Sci. **19**, 289 (2005).
[25] I. Eliazar and J. Klafter, J. Stat. Phys. **119**, 165 (2005).